

# Remote entanglement preparation

C. Spee,<sup>1</sup> J. I. de Vicente,<sup>1,2</sup> and B. Kraus<sup>1</sup>

<sup>1</sup>*Institute for Theoretical Physics, University of Innsbruck, Innsbruck, Austria*

<sup>2</sup>*Departamento de Matemáticas, Universidad Carlos III de Madrid, Leganés (Madrid), Spain*

We introduce a new multipartite communication scheme, with the aim to enable the senders to remotely and obliviously provide the receivers with an arbitrary amount of multipartite entanglement. The scheme is similar to Remote State Preparation (RSP). However, we show that even though the receivers are restricted to local unitary operations, the required resources for remote entanglement preparation are less than for RSP. In order to do so we introduce a novel canonical form of arbitrary multipartite pure states describing an arbitrary number of qubits. Moreover, we show that if the receivers are enabled to perform arbitrary local operations and classical communication, the required resources can drastically be reduced. We employ this protocol to derive robust entanglement purification protocols for arbitrary pure states and show that it can also be used for sending classical information.

Quantum information theory offers revolutionary ways to process and transmit information. One of its fundamental goals is to understand which classical and quantum resources are necessary to achieve certain tasks, like for instance quantum computation or quantum communication, e. g. transmission of quantum states. Naturally, entanglement is an important resource in this context. Due to that and its foundational interest an enormous effort has been made to achieve a better understanding of entanglement. However, despite considerable progress, many questions remain unanswered, particularly in the realm of multipartite states [1]. In this context, the following questions, which we address in this article, arise very naturally: Do there exist quantum communication protocols which allow some parties to supply other spatially separated parties with arbitrary multipartite entanglement instead of arbitrary states? If the answer is yes, what are the required resources and how do they compare to the ones required in other communication protocols? Our purpose is two-fold, besides shedding light in quantum information protocols and their cost, we aim at obtaining a better understanding of multipartite entanglement and its applications.

Questions as those posed above are the focus when addressing the limits of quantum information processing in fundamental protocols such as teleportation and Remote State Preparation (RSP). The aim of the latter is to enable the sender, Alice ( $A$ ), who is spatially separated from the receiver, Bob ( $B$ ), to prepare  $B$ 's system in a certain state known to  $A$  using entanglement and classical communication [2, 3]. The protocol is called faithful if  $B$  obtains deterministically the desired state and it is called oblivious if there is no leaking of further information about the state to  $B$  than the information that is already contained in the state itself. One distinguishes between the single copy case and the asymptotic case [3, 4], where the fidelity tends to unity, as the number of systems prepared tends to infinity. In the later case, it has been shown that one classical bit (cbit) and one ebit, i.e. one maximally entangled 2-qubit state, per qubit

sent is required to accomplish this task if the protocol is supposed to work for an arbitrary state [4]. In this article we will focus on the single copy case. There, it has been proven that the resources needed to perform RSP for a generic ensemble of states coincide with the resources required for quantum teleportation [6], i.e. 1 ebit and 2 cbits per transmitted qubit [2, 5]. Thus, the fact that  $A$  knows the state to be prepared does not help in reducing the required resources, unless one would restrict the set of states to be transmitted [2, 7].

In this paper we consider a different scenario. In contrast to RSP, the aim here is not to enable  $A$  to prepare an arbitrary state for  $B$ , but to provide him with an arbitrary amount of multipartite entanglement. Moreover, in contrast to RSP, all parties, the ones in  $A$  ( $A_i$ ) and the ones in  $B$  ( $B_i$ ) only need to act locally. We will show that this multipartite Remote Entanglement Preparation (REP) requires less resources than RSP, despite the fact that we restrict the action of  $A$  and  $B$  to be local. The classical communication within  $A$  and  $B$  resp. is for free.

The outline of the paper is the following. First of all, we introduce a novel Canonical Form (CF) for arbitrary  $n$ -qubit states. Any  $n$ -qubit state can be transformed into its CF via Local Unitary operators (LUs). This CF depends on  $P_n = 2^{n+1} + 2^{n-3} - 3(n+1)$  parameters and is based on the decomposition of arbitrary 3-qubit states introduced in [8]. Then, we use the notion of Deterministic Gate Implementation (DGI) [9] to show that any  $n$ -qubit state in the CF can be generated deterministically by measuring locally  $P_n$  qubits of a certain  $(n+P_n)$ -qubit resource state. More precisely, for any measurement outcome, the resulting  $n$ -qubit state is LU-equivalent to the desired state. These properties ensure that if  $A$  and  $B$  would share the  $(n+P_n)$ -qubit state, where  $A$  holds the  $P_n$  qubits to be measured, she is able to prepare  $B$ 's system in an arbitrary state in the CF up to LUs, which are performed by  $B$  after receiving the classical information of  $A$ . Since the classical information does not depend on the transmitted state, the protocol is oblivious. Moreover, it is faithful. The parties constituting

$B$ , knowing that they will receive a state in the CF can then, in case this is required apply LUs to obtain an arbitrary state. However, in most cases, this will, arguably not be required, since  $B$  will make use of the entanglement contained in the state. That is, the resource is the entanglement, not a particular state. We show that only  $(2n-1)/n$  cbits per qubit suffice to accomplish this task, which is less than is required in RSP [5]. Furthermore, we go one step beyond that and provide a protocol for remote maximally entangled state preparation. In this context we use the notion of Maximally Entangled Sets (MESs) [10]. Given a MES for  $n$ -partite states, any other  $n$ -partite state can be prepared deterministically via Local Operations and Classical Communication (LOCC). We show that in the case of 3-qubit states, this allows to reduce the required resources drastically. Moreover, we use the REP protocol to derive robust purification protocols for arbitrary states and to transmit classical information.

Throughout the paper  $W(\alpha)$  denotes the unitary  $e^{i\alpha W}$ , for  $W \in \{X, Y, Z\}$ , where  $X, Y, Z$  denote the Pauli operators, which will sometimes also be denoted by  $\sigma_i$  and  $|\pm\rangle$  denotes the eigenbasis of  $X$ . The subscripts of an operator, or a state denotes the system the operator is acting on or the system it is describing.  $U_{i_1, \dots, i_k}(\alpha)$  denotes the phase gate  $e^{i\alpha Z_{i_1} \otimes \dots \otimes Z_{i_k}}$  acting on the  $k$  systems  $i_1, \dots, i_k$ . The Bell basis is denoted by  $\{\sigma_i |\Phi^+\rangle\}_{i=0}^3$ , with  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  and bold letters denote a bitstring, e.g.  $\mathbf{i} \in \{0, 1\}^n$ .

Let us start by introducing a CF of arbitrary  $n$ -qubit states. In [8] we have shown that an arbitrary 3-qubit state is (up to LUs) of the form

$$|\Psi_3\rangle = U_{13}(\alpha)V_3U_{12}(\beta)V_2(\delta, \epsilon)U_{23}(\gamma)|+\rangle^{\otimes 3}, \quad (1)$$

where  $V_3 = X(-\frac{\pi}{4})Z(-\frac{\pi}{4})H$ ,  $V_2(\delta, \epsilon) = Y(\delta)Z(\epsilon)H$ , and  $H$  denotes the Hadamard gate [26]. Here, and in the following, the qubits are numerated consecutively.

In order to see how this CF can be generalized to more qubits, we consider first the 4 qubit case. W. l. o. g we write an arbitrary 4-qubit state (up to a LUs) as  $\frac{1}{\sqrt{2}}(|0\rangle|\Psi_3^0\rangle + |1\rangle U_2U_3U_4|\Psi_3^1\rangle)$ , where both,  $|\Psi_3^{0,1}\rangle$  are (normalized) 3-qubit states in the CF [Eq. (1)] and  $U_i = V_i^\dagger Z(2\gamma_i)V_i$ . Here, and in the following  $V_i$  denotes a unitary of the form  $Y(\beta_i)Z(\alpha_i)$  for some phases  $\alpha_i, \beta_i$ . Using the CF of the three qubit states and the fact that  $|0\rangle_1 \langle 0| \otimes U_{23}(\alpha) + |1\rangle_1 \langle 1| \otimes U_{23}(\beta) = U_{123}(\frac{\alpha-\beta}{2})U_{23}(\frac{\alpha+\beta}{2})$ , it is straightforward to find the following CF for 4-qubit states,

$$|\Psi_4\rangle = \prod_{i=2}^4 U_{1i}(\gamma_i)V_iU_{124}(\tilde{\alpha})U_{24}(\alpha')U_{123}(\tilde{\beta})U_{23}(\beta')X_3(\frac{\pi}{4}) \\ U_{13}(\tilde{\delta})X_3(-\frac{\pi}{4})V_3'U_{13}(\tilde{\epsilon})H_3U_{134}(\tilde{\gamma})U_{34}(\gamma')|+\rangle^{\otimes 4},$$

where  $V_3'$  is defined as  $V_i$  above. In the same way the CF

of an arbitrary number of qubits,  $n$ , can be derived and we find

$$|\Psi_n\rangle = \prod_{j=2}^n U_{1j}(\gamma_j)V_j \left( \frac{1}{\sqrt{2}}(|0\rangle|\Psi_{n-1}^0\rangle + |1\rangle|\Psi_{n-1}^1\rangle) \right), \quad (2)$$

where  $|\Psi_{n-1}^{0,1}\rangle$  denote  $(n-1)$ -qubit states in the CF. Thus, the CF of  $n$ -qubit states is obtained by the CF of  $(n-1)$ -qubit states and is therefore defined via the CF of 3-qubit states [Eq. (1)]. Hence, the CF can be easily described and constructed for arbitrary  $n$ . The number of parameters required in the CF of an  $n$  qubit state,  $P_n$ , is given recursively by  $2P_{n-1} + 3(n-1)$ , with  $P_3 = 5$ , which leads to  $P_n = 2^{n+1} - 3(n+1) + 2^{n-3}$ . Note that the diagonal elements of the reduced state of the first qubit are equal.

Let us now show, how an arbitrary  $n$ -qubit state in the CF can be generated deterministically by measuring  $P_n$  qubits of a  $(n+P_n)$ -qubit resource state. First we show how an arbitrary gate can be implemented deterministically (see also [9, 11, 12]). In [9] it has been shown how arbitrary phase gates can be implemented deterministically on an arbitrary unknown input state,  $|\phi\rangle$ . We briefly recall the idea here and then generalize it to arbitrary DGI.

*DGI of phase gates [9]:* To implement the single qubit gate,  $Z(\alpha)$  deterministically one might use the GHZ-state,  $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$  as resource state and perform a Bell measurement on one of the qubits, say qubit 3 and the input qubit. Depending on the measurement outcome,  $i$ , party 2 chooses  $\alpha_j$  to be  $+\alpha$  for  $i = 0, 3$  and  $-\alpha$  for  $i = 1, 2$  and measures in the orthogonal basis

$$\mathcal{B}_{\alpha_j} = \{|\phi^i(\alpha_j)\rangle \equiv Z^i Z(-\alpha_j)|+\rangle\}_{i=0,1}. \quad (3)$$

It is easy to see that if party 2 measures  $|\phi^k(\alpha_j)\rangle$ , the resulting state is  $Z^k \sigma_i U(\alpha)|\phi\rangle$ , where  $i$  ( $k$ ) denotes the measurement outcome of party 3 (2) resp.. Thus, choosing the measurement basis of party 2 depending on the outcome of the Bell-measurement allows to implement the gate deterministically, since the correction operator,  $Z^k \sigma_i$  can be applied to obtain the desired state. In the following we will call the qubit, which has to be measured in the basis  $\mathcal{B}_{\alpha_j}$  in order to implement the gate, controlling qubit. The resource state to implement a two-qubit phase gate  $U_{12}(\alpha)$  on an arbitrary input state,  $|\phi\rangle_{b_1, b_2}$ , is  $\frac{1}{\sqrt{2}}[|0\rangle(|0000\rangle + |1111\rangle) + |1\rangle(|0011\rangle + |1100\rangle)]$ . Here, qubit 1 is the controlling qubit. First, Bell measurements on qubit 2 and  $b_1$  and one on 4 and  $b_2$ , with outcome  $i_1, i_2$  resp. are performed. Similarly to before, the controlling qubit is measured in the basis,  $\mathcal{B}_{\alpha_j}$  [Eq. 3], with  $\alpha_j = +\alpha$  if  $[U(\alpha), \sigma_{i_1} \otimes \sigma_{i_2}] = 0$  and  $\alpha_j = -\alpha$  if  $U(\alpha)\sigma_{i_1} \otimes \sigma_{i_2} = \sigma_{i_1} \otimes \sigma_{i_2} U(-\alpha)$ . Measuring  $|\phi^k(\alpha_j)\rangle$  leads to the output state  $\sigma_z^{k i_1} \otimes \sigma_z^{k i_2} U(\alpha)|\phi\rangle$ . Note that the correction operators are local. Similarly, gates

of the form  $U_m(\alpha) = e^{i\alpha\sigma_z^{\otimes m}}$  can be implemented requiring only one controlling qubit, which is again measured in the basis  $\mathcal{B}(\alpha_j)$  for properly chosen  $\alpha_j = \pm\alpha$ . Depending on the measurement outcome the gate  $\sigma_z^{\otimes m}$  might occur in the correction operators. Note that each parameter is fixed by measuring one controlling qubit.

*DGI for arbitrary gates:* Let us first derive the DGI of an arbitrary single qubit unitary,  $U(\alpha^1, \alpha^2, \alpha^3) \equiv Z(\alpha^1)HZ(\alpha^2)HZ(\alpha^3)$ . The three gates could be implemented consecutively using the resource state  $|GHZ\rangle$ , and  $H_1H_2\mathbb{1}_3|GHZ\rangle$ , respectively. However, the Bell measurements required to map the output of the first gate to the input of the second gate, can be performed independently of the input state. We call the application of this projection onto the state  $|\Phi^+\rangle$  connection between the resource states. Connecting the three resource states above leads to the 6-qubit resource state  $\langle\Phi^+|_{c_1,b_2}\langle\Phi^+|_{c_2,b_3}H_{b_2}H_{c_2}\otimes_{i=1}^3|GHZ\rangle_{a_i,b_i,c_i}$ . Similarly to before, the measurement bases for the three controlling qubits are  $\mathcal{B}_{\alpha_{j_m}^m}$  for  $m = 1, 2, 3$ , where  $\alpha_{j_m}^m$  is either  $+\alpha^m$  or  $-\alpha^m$ , depending on the outcome of the previous measurements. Note that the order of the unitaries has to be respected. Note further that this concatenation works, because the only gates which occur here are either Pauli rotations or Clifford gates. Thus, any gate occurring due to the measurement either maps the phase gate into itself, or into its adjoint. Using that an arbitrary two-qubit gate can be written as  $U = U_{loc}U_d\tilde{U}_{loc}$  where  $\tilde{U}_{loc}$  and  $U_{loc}$  are local unitaries and the non-local part,  $U_d$ , is given by  $U_d = e^{i\alpha\sigma_z^1\otimes\sigma_z^2}e^{i\beta\sigma_x^1\otimes\sigma_x^2}e^{i\gamma\sigma_y^1\otimes\sigma_y^2}$  [13], it is straightforward to determine the resource state required to implement this gate. First, one decomposes  $U$  in terms of the Clifford gates,  $H$  and  $X(\frac{\pi}{4})$ , and single- and two-qubit phase gates. Then, for the successive implementation of the gates the resource states for the phase gates need to be connected in the correct order. As before, the signs of the measurement angles depend on the outcomes of the previous measurements. Since any gate can be decomposed into single- and two-qubit gates, the successive DGI explained here, allows to deterministically implement an arbitrary gate. The corresponding resource states are always stabilizer states. This is due to the fact that the resource states for the phase gates are stabilizer states, the fixed local gates are Clifford gates and the connection, i.e. the projection onto  $|\Phi^+\rangle$ , transforms two stabilizer states again into a stabilizer state.

We combine now the CF of multipartite states and the DGI to derive a REP protocol, where  $A$ 's aim is to prepare an arbitrary  $n$ -qubit state,  $|\Psi\rangle$ , in the CF for  $B$ , with the following properties: (i) the protocol is oblivious (ii) it is faithful (iii) all actions performed by  $A$  and  $B$  are local (iv) the number of ebits shared by  $A$  and  $B$  is  $n$  (v) the number of cbits needed to be communicated from  $A$  to  $B$  is  $2n - 1$ . Note that  $B$  has to know that he receives a state in the CF since otherwise his state would be completely mixed to the randomness of the LUs. How-

ever, the description of the CF is a side information that is independent of  $n$  and it can be agreed upon by  $A$  and  $B$  beforehand as part of the protocol. Thus, even though  $B$  receives an arbitrary amount of multipartite entanglement, this multipartite protocol outperforms previously known RSP despite the fact that all actions are local single qubit operations.

To derive the REP protocol, we denote by  $U_n$  the unitary such that  $|\Psi_n\rangle \equiv U_n|+\rangle^{\otimes n}$  is the CF of an  $n$ -qubit state and by  $|\tilde{\Phi}_n\rangle$  the corresponding resource state. Since  $U_n$  can be decomposed into local Clifford gates and  $P_n$  phase gates,  $P_n$  controlling qubits are required. Using  $|+\rangle^{\otimes n}$  as input state, leads to the  $(n + P_n)$ -qubit resource state,  $|\Phi_n\rangle \equiv \otimes^n \langle + | \tilde{\Phi}_n \rangle$ . Measuring locally the  $P_n$  controlling qubits leads to one of the states  $\sigma_{i_1} \otimes \dots \otimes \sigma_{i_n} |\Psi_n\rangle$  for  $i_j \in \{0, \dots, 3\}$  for  $j \geq 2$  and  $i_1 \in \{0, 3\}$ . REP can thus be achieved if  $A$  and  $B$  share the state  $|\Phi_n\rangle$ , where  $A$  is holding all controlling qubits and  $B$  the other  $n$  qubits. Due to the fact that  $i_1 \in \{0, 3\}$  only  $2n - 1$  cbits are required to learn the local correction operators. The protocol is faithful and oblivious since the correction operators,  $\sigma_{i_1} \otimes \dots \otimes \sigma_{i_n}$ , are independent of the prepared state [27].

Considering the first non-trivial example of generation of arbitrary bipartite entanglement, we choose the CF  $|\Psi\rangle = U_{12}(\alpha)|+\rangle^{\otimes 2}$  for  $\alpha \in [0, \frac{\pi}{2}]$ . REP can be achieved if  $A$  and  $B$  share the GHZ-state,  $|GHZ\rangle = \frac{1}{\sqrt{2}}(|++\rangle_{A_1,B_1,B_2} + |--\rangle_{A_1,B_1,B_2})$ , and  $A$  performs a measurement on her qubit,  $A_1$ , in the basis  $\mathcal{B}_\alpha$ . Depending on the measurement outcome,  $B$ 's system is either prepared in the state  $|\Psi\rangle$  or in the state  $\sigma_z \otimes \sigma_z |\Psi\rangle$ . Once  $A$  tells  $B$  her measurement outcome,  $B$  can locally apply the Pauli operator to obtain the state  $|\Psi\rangle$  deterministically. The protocol is oblivious since the transmitted classical information is independent of the prepared state. Since the classical information within  $B$  is for free, the required resources are  $\frac{1}{2}$  ebits and  $\frac{1}{2}$  cbits per transmitted qubit. In contrast to that, RSP requires 1 ebit and 2 cbits per qubit [5].

The generalization of this protocol to three and more qubits is straightforward. Considering for instance the 3-qubit case, it is lengthy, but straightforward to show that the resource state is given by the 8 qubit state,

$$|\Phi_3\rangle = H_6Z_8(\frac{\pi}{4})H_8Z_7(-\frac{\pi}{4})Z_2(\frac{\pi}{4})W_{12}W_{23} \quad (4)$$

$$W_{37}W_{24}W_{25}W_{27}W_{46}W_{48}W_{58}W_{67}W_{78}|+\rangle^{\otimes 8},$$

where  $W_{ij} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes Z$ . The qubits 6, 7 and 8 are held by  $B$ . Whereas, the first five qubits are the controlling qubits, which are held by  $A$ . She chooses freely the parameters in the decomposition Eq. (1), i.e. first, qubit 1 is measured in  $\mathcal{B}_\gamma$  (see Eq. (3)), then qubit 2 in  $\mathcal{B}_\epsilon$  etc., and therefore also the multipartite entanglement  $B$  obtains. As before, the measurements have to be performed in a certain order to make sure that the protocol

is deterministic.  $A$  computes the local correction operators and sends it to  $B$ . This requires only  $\frac{5}{3}$  cbits per qubit, i.e. in total 1 cbit less than in RSP. The reason for that is again that on qubit 6 only one of the two Pauli operators,  $\mathbb{1}, \sigma_z$  can occur. The required entanglement shared between  $A$  and  $B$  corresponds to 1 ebit per qubit.

So far we have considered the situation where  $A$  wants to provide  $B$  with an arbitrary amount of multipartite entanglement. The local unitary operations, can be implemented by  $B$ , in case this is really necessary. However, one might even go beyond that as our only requirement is that all the actions performed by  $A$  and especially by  $B$  are local. Suppose that we allow  $B$  to perform an arbitrary LOCC protocol. In this case it is sufficient to provide  $B$  with those states, which are required to obtain any other state via LOCC. Let us consider the simplest case of bipartite entanglement. If  $A$  provides  $B$  with the maximally entangled state,  $|\Phi^+\rangle$ ,  $B$  can afterwards transform this state deterministically into any other state [14]. Since this can be achieved by sharing initially a GHZ-state, the required resources are  $\frac{1}{2}$  ebits and  $\frac{1}{2}$  cbits per qubit, which coincides in this case with the resources required for REP. Let us now show that the resources can be drastically reduced in case of three-partite entanglement. In [10] we introduce the concept of Maximally Entangled Sets (MESs). A MES,  $S$ , is a set of states with the following properties: (i) No state in  $S$  can be obtained from any other state in  $S$  via LOCC and (ii) for any  $n$ -qubit state,  $|\Psi\rangle$ , there exists a state in  $S$  such that  $|\Psi\rangle$  can be obtained from it via LOCC. Thus, the set is maximally entangled. Whereas in the bipartite case,  $S$  contains only one maximally entangled state, for more qubits a whole set of states is required. For instance in the three-qubit case we show in [10] that any state in  $S_3$  is of the form given in Eq. (1) with  $\delta = \epsilon = \pi/4$ . Thus, the corresponding resource state can be determined by measuring on the 8-qubit state [Eq. (4)] the qubits 2 and 3 in the  $\sigma_y$  basis, leading to a 6-qubit stabilizer state (see Appendix B, Fig. 2 a)). If  $A$  and  $B$  share this entangled state,  $A$  can remotely and obviously prepare any state in  $S_3$  for  $B$  by choosing appropriate measurements on the 3 controlling qubits. Notice that the reduced state of  $B$ 's qubits is the same as before, so they still need to share 1 ebit per qubit of entanglement. However,  $A$  just needs to send three cbits to  $B$  as she only needs to specify the outcomes of three local Von Neumann measurements. That is, in order to achieve remote maximal entanglement preparation the amount of classical communication can be drastically reduced to 1 cbit per qubit.

Once the  $B_i$ 's have the state  $|\Psi\rangle$ , they can use it to implement via LOCC probabilistically the possibly non-local map,  $\mathcal{E}(\cdot) = A_\Psi \cdot A_\Psi^\dagger$ , for some operation  $A_\Psi$  [15]. By restricting the transmitted classical information,  $A$  can control this capability of  $B$ . In fact, it is easy to see that if  $A$  would send  $B$  only partial information about the correction operations, they can only imple-

ment maps of the form  $\sum_i A_\Psi \sigma_i \cdot \sigma_i A_\Psi^\dagger$ , where the sum is over all qubits whose correction operators are unknown to  $B$ . For instance, if  $A$  does not tell  $B$  the correction operator of the first qubit,  $B$  can only realize the map  $\sum_{i=0}^3 A_\Psi \sigma_i^1 \rho \sigma_i^1 A_\Psi^\dagger = \sum_{i=0}^3 A_\Psi \mathbb{1}_1 \otimes \rho_{-1} A_\Psi^\dagger$ , for an arbitrary input state  $\rho$ . Here,  $\rho_{-1} = \text{tr}_1(\rho)$  [28]. Thus, our protocol does not only enable  $A$  to have control over the entanglement the  $B_i$ 's share but also over the operations they can implement with the prepared state via restricting the classical communication.

This protocol can also be used to send classical information. To see that we consider a subclass of the locally maximally entangleable states (LMESs) [16], which are states that can be generated (up to LUs) by applying phase gates to  $|+\rangle^{\otimes n}$ . The subclass of interest is chosen such that the generalized stabilizer [16] is generated by  $S_j = X_j \otimes U_j$ , where  $U_j$  is a phase gate acting on the neighborhood of qubit  $j$ , i.e. on all the qubits which are coupled to qubit  $j$ . Since  $S_l Z^i |\Psi\rangle = (-1)^{il} Z^i |\Psi\rangle \forall l$ , the set  $\{Z^i |\Psi\rangle\}_{i \in \{0,1\}^n}$  is the common eigenbasis of the stabilizer. In Appendix A we show that if  $|\Psi\rangle$  is known, the expectation value  $\langle S_j \rangle$  can be determined by measuring locally  $X_j$  and  $Z_k$ , for all qubits  $k$  in the neighborhood of qubit  $j$ . Especially for so-called regular LMES (see Appendix A and [17]) several cbits can be determined in this way from a single copy of the state. The procedure to send classical information is then as follows.  $A$  uses REP to prepare a LMES,  $|\Psi\rangle$ . Depending on her measurement outcomes,  $B$  obtains with equal probability one of the states  $\{Z^i |\Psi\rangle\}_{i \in \{0,1\}^n}$ , where the bit-string  $i$  is known by  $A$  [29]. If  $A$  and  $B$  agree beforehand on which LMES basis to use as part of the protocol,  $B$  can perform local measurements to determine the bits  $\langle S_j \rangle$  for some values of  $j$ .

As a last application we show that REP can also be used to derive robust entanglement purification protocols for arbitrary states. Those protocols are very desirable since most applications of quantum information theory require pure entangled states. Despite its importance, purification protocols [18] have been only devised for stabilizer states [19], the W state [20] and LMESs [17]. In Appendix B we show that the resource state of our REP behaves basically like a two-colorable graph state for which efficient purification procedures have been presented [19, 21]. Thus, an arbitrary state can be purified by purifying the resource state and then performing the local measurements which lead to the desired pure target state. In Appendix B we show that this purification process outperforms previously known multipartite purification protocols in the sense that the tolerable noise is much higher.

In summary, we have introduced a multipartite communication scheme, REP, which, in the spirit of teleportation and RSP, allows a party to prepare arbitrary forms of multipartite entanglement for other spatially separated parties. We have also shown that REP can be achieved



consuming less resources than RSP even though the operations the parties can do are more limited. Finally, we have introduced several applications of the protocol. In the future, it will be interesting to study whether both the classical and quantum resources consumed by our REP protocol are optimal or if they can be further reduced.

We would like to thank W. Dür for fruitful discussions. The research was funded by the Austrian Science Fund (FWF): Y535-N16.

- 
- [1] See e. g. the reviews M.B. Plenio and S. Virmani, *Quantum Inf. Comput.* **7**, 1 (2007); R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
  - [2] H.-K. Lo, *Phys. Rev. A* **62**, 012313 (2000).
  - [3] C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, *Phys. Rev. Lett.* **87**, 077902 (2001).
  - [4] C. H. Bennett, P. Hayden, D. W. Leung, P. W. Shor and A. Winter, *IEEE Trans. Inform. Theory* **51**, 56 (2005).
  - [5] D. W. Leung and P. W. Shor, *Phys. Rev. Lett.* **90**, 127905 (2003).
  - [6] C. H. Bennett, G. Brassard, C. Crépeau, R. Josza, A. Peres, and W. K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).
  - [7] A. K. Pati, *Phys. Rev. A* **63**, 014302 (2000).
  - [8] J.I. de Vicente, T. Carle, C. Streitberger, and B. Kraus, *Phys. Rev. Lett.* **108**, 060501 (2012).
  - [9] W. Dür, M. J. Bremner and H. J. Briegel, *Phys. Rev. A* **78**, 052325 (2008).
  - [10] J.I. de Vicente, C. Spee and B. Kraus, in preparation.
  - [11] J. Anders, D. K. L. Oi, E. Kashefi, D. E. Browne and E. Andersson, *Phys. Rev. A* **82**, 020301(R) (2010).
  - [12] R. Raussendorf and H.J. Briegel, *Phys. Rev. Lett.* **86**, 5188–5191 (2001).
  - [13] B. Kraus and J. I. Cirac, *Phys. Rev. A* **63**, 062309 (2001).
  - [14] M. Nielsen, *Phys. Rev. Lett.* **83**, 436 (1999).
  - [15] A. Jamiolkowski, *Rep. Math. Phys.* **3**, 275 (1972); J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, *Phys. Rev. Lett.* **86**, 544 (2001).
  - [16] C. Kruszynska and B. Kraus, *Phys. Rev. A* **79**, 052304 (2009).
  - [17] T. Carle, B. Kraus, W. Dür and J. I. de Vicente, *Phys. Rev. A* **87**, 012328 (2013).
  - [18] See e. g. the review W. Dür and H.J. Briegel, *Rep. Prog. Phys.* **70**, 1381 (2007).
  - [19] W. Dür, H. Aschauer, and H.J. Briegel, *Phys. Rev. Lett.* **91**, 107903 (2003); H. Aschauer, W. Dür, and H.J. Briegel, *Phys. Rev. A* **71**, 012319 (2005).
  - [20] A. Miyake and H.J. Briegel, *Phys. Rev. Lett.* **95**, 220501 (2005).
  - [21] C. Kruszynska, A. Miyake, H.J. Briegel, and W. Dür, *Phys. Rev. A* **74**, 052316 (2006).
  - [22] R. Raussendorf, D. E. Browne and H.J. Briegel, *Phys. Rev. A* **68**, 022312 (2003).
  - [23] M. Hein, J. Eisert, and H.J. Briegel, *Phys. Re. A* **69**, 062311 (2004).
  - [24] A. Cabello, A.J. López-Tarrida, P. Moreno, and J.R. Por-

tillo, *Phys. Lett. A* **373**, 2219 (2009).

- [25] A. Cabello, A.J. López-Tarrida, P. Moreno, and J.R. Por-tillo, *Phys. Rev. A* **80**, 012102 (2009).
- [26] In fact, in [8] we have shown that an arbitrary 3-qubit state is (up to LUs) of the form  $|\Psi\rangle = U_{12}U_{13}|+\rangle|\psi_s\rangle$ , where  $|\psi_s\rangle = a|00\rangle + b|11\rangle$ ,  $a \leq b$  has Schmidt decomposition and  $U_{1j} = |0\rangle_1\langle 0| \otimes \mathbf{1}_j + |1\rangle_1\langle 1| \otimes U_j$  with  $U_2 = Z(\alpha)Y(\beta)Z(\gamma)$  and  $U_3 = Y(\beta')$ . Using that  $U_2$  can be written as  $W^\dagger Z(-2\beta)W$  with  $W = Y(\delta)Z(\epsilon)$  and  $|\psi_s\rangle$  is proportional to  $(H_2 \otimes Z_3(-\frac{\pi}{4})H_3)e^{i\gamma\sigma_z^2\otimes\sigma_z^3}|+\rangle^2$  one derives Eq. (1).
- [27] Note that  $A$  could also implement arbitrary LUs on all qubits, but the first, requiring the same resources.
- [28] In order to have also for the first qubit all possible correction operations, we use a more symmetrized CF, namely  $1/\sqrt{2}(|0\rangle|\Phi\rangle + |1\rangle\sigma_x^1|\Phi\rangle)$ , instead of  $|\Phi\rangle$ , where  $A$  measures qubit 1 in the computational basis.
- [29] Note that without knowing  $\mathbf{i}$   $B$  system is completely mixed, even if he knows  $|\Psi\rangle$ .

## APPENDIX A: SENDING CLASSICAL INFORMATION

The REP protocol does not only allow us to transmit quantum information but also classical information. In order to send classical information one restricts to REP of LMESs [16]. Let us first review some of the basic properties of LMESs. Any  $n$ -qubit LMES can be written up to LUs in the form:

$$|\Psi\rangle \equiv U|+\rangle^n = U_{1,\dots,n} \prod U_{i_{k_1},\dots,i_{k_n}} \dots \prod U_i |+\rangle^{\otimes n}, \quad (5)$$

where  $U_{i_{k_1},\dots,i_{k_n}} = e^{i\alpha_{i_{k_1},\dots,i_{k_n}}\sigma_z^{i_{k_1}}\otimes\dots\otimes\sigma_z^{i_{k_n}}}$ . Note that equivalently a LMES (up to LUs) can be written as  $2^{-\frac{n}{2}} \sum_{i_1,\dots,i_n=0}^1 e^{i\beta_{i_1\dots i_n}} |i_1\dots i_n\rangle$ , with  $\beta_{i_1\dots i_n} \in \mathbf{R}$ . In the following we will call a state with  $\beta_{i_1\dots i_n} \in \{0, \pi\}$   $\pi$ -LMES. It has been shown that for any  $n$ -qubit LMES,  $|\Psi\rangle$  [Eq. (5)], one can construct a generalized stabilizer, generated by  $n$  operators  $S_k, k \in \{1, \dots, n\}$  [16]. These hermitian and unitary operators are of the form  $S_k = UX_kU^\dagger$  and their common eigenbasis is given by  $\{Z^{\mathbf{i}}|\Psi\rangle\}_{\mathbf{i} \in \{0,1\}^n}$ , i.e.

$$S_k Z^{\mathbf{i}} |\Psi\rangle = (-1)^{i_k} Z^{\mathbf{i}} |\Psi\rangle \quad \forall \mathbf{i}. \quad (6)$$

A LMES is called  $k$ -colourable if one can assign to every qubit one out of  $k$  colours in such a way that no qubits interacting via a phase gate have the same colour [17].

LMESs can be easily implemented via DGI, since  $U$  [Eq. (5)] corresponds to a product of phase gates. As has been shown in the main text, the correction operators that can occur due to the implementation procedure are always of the form  $Z^{\mathbf{i}}$  for  $\mathbf{i} \in \{0,1\}^n$ . In particular in the DGI of a general LMES of the form of Eq. (5) all possible Pauli corrections occur with the same probability. This is easy to see since in DGI of a single-qubit

unitary  $U_l$  the correction  $\mathbb{1} \otimes \dots \otimes Z_l \otimes \dots \mathbb{1}$  occurs with probability  $\frac{1}{2}$ , which ensures that in total the bitstring  $\mathbf{i}$  corresponding to the correction operation  $Z^{\mathbf{i}}$  will be random.

In order to transmit classical information,  $A$  remotely prepares a LMES,  $|\Psi\rangle$  (known to  $B$ ), for  $B$ . So  $B$  obtains one of the states  $Z^{\mathbf{i}}|\Psi\rangle$ . Note that the bitstring  $\mathbf{i}$  only depends on  $A$ 's measurement results. Therefore, it is completely random and known to  $A$ . We show next that, once  $B$  receives a copy of the state  $Z^{\mathbf{i}}|\Psi\rangle$ , for some  $\mathbf{i} \in \{0,1\}^n$ , he can perform local measurements to obtain partial information on the bitstring. The classical information that is sent corresponds to the part of the bitstring that  $B$  can determine via local measurements. In order to show that  $B$  can get partial information on the bitstring, we make use of the following lemma.

**Lemma 1.** *Let  $|\Psi_0\rangle$  be a  $\pi$ -LMES, so that  $\{|\Psi_{\mathbf{i}}\rangle = Z^{\mathbf{i}}|\Psi_0\rangle\}_{\mathbf{i} \in \{0,1\}^n}$  is the common eigenbasis of a generalized stabilizer generated by  $\{S_j = X_j \otimes U_j\}_{j=1}^n$ . Then for any  $\mathbf{k} \in \{0,1\}^{n-1}$  (denoting the computational basis of all qubits but qubit  $j$ ) and  $|l_x^j\rangle \in \{|+\rangle, |-\rangle\}$  with  $\langle l_x^j | \Psi_{\mathbf{i}} \rangle \neq 0$  it follows that  $\langle \Psi_{\mathbf{i}} | S_j | \Psi_{\mathbf{i}} \rangle = \langle l_x^j | \mathbf{k} | S_j | l_x^j \rangle$ .*

*Proof.* Let  $|\Psi\rangle$  be an element of the common eigenbasis. We write  $|\Psi\rangle = \sum_{\mathbf{k} \in \{0,1\}^n} |\mathbf{k}\rangle \langle \mathbf{k} | \Psi \rangle \equiv 2^{n-1/2} \sum_{\mathbf{k} \in \{0,1\}^{n-1}} |\mathbf{k}\rangle |\phi_{\mathbf{k}}\rangle$ , with  $|\phi_{\mathbf{k}}\rangle$  normalized. Due to the fact that  $|\Psi_0\rangle$  is a  $\pi$ -LMES, it can be easily seen that  $|\phi_{\mathbf{k}}\rangle \in \{|+\rangle, |-\rangle\} \forall \mathbf{k}$ . Then  $\langle \Psi | S_j | \Psi \rangle = 2^{-(n-1)} \sum_{\mathbf{k} \in \{0,1\}^n} \langle \phi_{\mathbf{k}} | X^j | \phi_{\mathbf{k}} \rangle \langle \mathbf{k} | U^j | \mathbf{k} \rangle$ , where we used the fact that  $U_j$  is diagonal. Using that the expectation value of  $S_j$  is  $\pm 1$  and the fact that  $|\langle \mathbf{k} | U^j | \mathbf{k} \rangle| = 1$  the equation above can only be fulfilled if  $\langle \phi_{\mathbf{k}} | X^j | \phi_{\mathbf{k}} \rangle \langle \mathbf{k} | U^j | \mathbf{k} \rangle = \langle S_j \rangle$  for any  $\mathbf{k}$ . Since furthermore  $\langle \phi_{\mathbf{k}} | Z | \mathbf{k} | \Psi_{\mathbf{i}} \rangle = 0 \forall \mathbf{k}$  this proves the statement.  $\square$

This Lemma allows us to identify the procedure for  $B$  to learn part of the bitstring. To determine  $\langle S_j \rangle$  measure  $X$  on particle  $j$  and  $Z$  on the particles  $N_j$  that are interacting with  $j$ . Denote by  $M_j$  all particles that are neither  $j$  nor in  $N_j$ . The state after the measurement corresponds to  $|l_x^j\rangle_j \otimes |\mathbb{1}\rangle_{N_j} \otimes |\phi^{l_x, \mathbb{1}}\rangle_{M_j}$  for  $|\phi^{l_x, \mathbb{1}}\rangle$  some state describing the remaining qubits and  $\mathbb{1} \in \{0,1\}^{|N_j|}$ . Note that we can choose, without loss of generality,  $|l_x^j \mathbb{1} \dots \mathbb{1}\rangle$  to compute the expectation value according to Lemma 1, since  $\langle l_x^j \mathbb{1} \dots \mathbb{1} | \Psi_{\mathbf{i}} \rangle \neq 0$  and  $S_j$  only acts on particle  $j$  and the qubits interacting with it. Using Lemma 1 the expectation value of  $X_j \otimes U_j$  is given by  $m_j m_{\mathbb{1}}$ , where  $m_j$  is the measurement outcome of party  $j$  and  $m_{\mathbb{1}}$  corresponds to  $\langle \mathbb{1} | U_j | \mathbb{1} \rangle$ . This procedure requires that  $B$  knows the generators of the generalized stabilizer of the LMES,  $|\Psi\rangle$ , that was prepared in DGI, i.e. he has to know  $|\Psi\rangle$ . Note that as long as the qubits  $j_i$  are not interacting with each other this procedure allows to determine all  $\langle S_{j_i} \rangle$  on a single copy of the state. Note further that the

measurements are local.

A  $\pi$ -LMES which is  $k$ -colourable and only involves  $k$ -qubit interactions is called  $k$ -regular LMES [17]. It is easy to see that they fulfill the above stated condition and, therefore, the expectation values corresponding to the same colour can be evaluated on a single copy. Hence,  $k$ -regular LMESs are useful in this context. This kind of classical information might be useful in the context of secure multipartite communication.

The resource state for the remote preparation of a LMES [Eq. (5)] is maximally entangled in the splitting  $A$  versus  $B$ , i.e. the reduced density matrix of  $B$  is totally mixed and the entanglement between  $A$  and  $B$  is 1 ebit per qubit. Despite that fact, we have seen that  $B$  can learn part of the bitstring via local measurements knowing that  $A$  has performed her measurements beforehand. As we have seen via choosing between different set-ups the REP protocol allows for the transmission of either quantum or classical information without actually sending it.

## APPENDIX B: ROBUST ENTANGLEMENT PURIFICATION TO ARBITRARY MULTIPARTITE STATES

While most applications in quantum information theory require entangled pure states, actual implementations unavoidably introduce noise to the target states. For this reason, purification protocols to transform locally several copies of a mixed noisy state into a more pure state have been thoroughly studied [18]. However, with the sole exceptions of the  $W$ -state [20] and the LMESs [17], purification protocols have been only devised for stabilizer states, which are states that are LU-equivalent to graph states. Let us briefly recall the definition of graph states. Graph states correspond to  $\pi$ -LMESs, where each phase gate is acting only on 2 qubits. They are often associated with a mathematical graph consisting of a set of vertices,  $V$ , representing the qubits, and a set of edges,  $E$ , representing the two-qubit phase gates [22]. Graph states constitute a notable class, for which efficient purification procedures have been given [19, 21]. However, as is easy to understand, not every mixed state can be purified locally and purification protocols can only succeed if the noise lies below a certain threshold.

Since our REP scheme relies on the parties sharing a stabilizer state and then making local measurements on it to obtain an arbitrary entangled state, this readily provides a purification protocol to *all* entangled states. Furthermore, it turns out that the graph states corresponding to our resource states fall under a class of graph states for which the existing purification protocols are particularly robust against noise. Thus, we will show that in certain scenarios using our scheme would allow to purify

to any given target state with significantly larger noise thresholds than the purification protocols particularly devised for other states such as the 3-qubit  $W$ -state [20] or LMESs [17].

A usual scenario when studying purification schemes is the following: A provider can prepare any desired entangled state, which he then sends to the different distant parties that need to use it. As each qubit is sent, it undergoes a local quantum channel which is considered to be the main source of noise. We will consider here that every qubit  $i$  that is sent is subjected to local depolarizing noise,

$$\mathcal{E}_p(\rho_i) = p\rho_i + \frac{1-p}{4}(\rho_i + \sigma_x\rho_i\sigma_x + \sigma_y\rho_i\sigma_y + \sigma_z\rho_i\sigma_z). \quad (7)$$

We assume that the noise level  $p$  is the same for all qubits that need to be transmitted. Once the distant parties receive the noisy state they can use LOCC to purify to their target state. To see more clearly how our protocol would work, let us restrict ourselves to the case of 3-qubit states for simplicity. Instead of sending the target state, A prepares the 8-qubit graph state corresponding to the resource stabilizer state given in Eq. (4) in the main text, which is needed to prepare any pure 3-qubit entanglement and sends the corresponding 3 qubits to the parties that constitute  $B$ . These qubits undergo then some noise, so  $A$  and  $B$  implement the corresponding graph-state purification protocol to recover the original resource state (4). After that,  $A$  carries out the required measurements and transmits the outcomes to the parties in  $B$ , who will then have a more pure copy of the target state of choice. Therefore, one just needs to study how robust is our graph state with the already devised purification procedures for this kind of states.

It turns out that one of the most relevant properties of graph states regarding purification is their colorability. A graph is 2-colorable if we can label each vertex using just two colors (say  $\mathcal{A}$  and  $\mathcal{B}$ ) with the rule that no adjacent vertices get the same color.  $N$ -colorable graphs are defined analogously. Purification protocols for 2-colorable graph states are simpler and usually more robust against noise (this is because the qubits with different colors are purified independently and as one purifies certain color, noise is introduced in the qubits of a different color). As one can easily see from Fig.1, our 8-qubit resource graph state is 3-colorable.

However, one may wonder whether there exists a 2-colorable LU equivalent state to which the parties can purify and after that apply local unitaries to obtain the relevant graph state. All the different classes of graph states up to 8 qubits have been studied in [23, 24]. Using the invariants given in [25], it is lengthy but straightforward to check that our graph belongs to the class number 98 in [24]. According to this reference, there is no 2-colorable graph in this class. Hence, one must use a 3-color purification procedure in our case. However, as

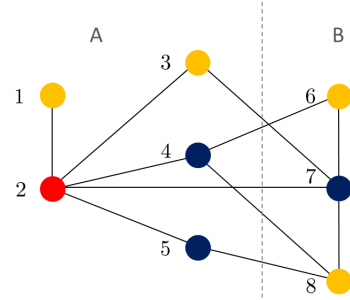


FIG. 1: This graphic shows the graph state that is LU-equivalent to the 8-qubit resource state for REP in the 3-qubit case.

depicted in Fig.1, one can color the qubits in such a way that the third color  $\mathcal{C}$  only appears once and not in the qubits that are sent to  $B$ . It turns out that if the noise only affects the qubits that are sent to  $B$ , the qubit with color  $\mathcal{C}$  does not get affected by the noise and, thus, a 2-colorable purification procedure purifying qubits  $\mathcal{A}$  and  $\mathcal{B}$  is enough to achieve the task. Implementing the procedure given in [19], one finds that the state can be purified if the noise affecting the 3 transmitted qubits is such that  $p \geq 0.39$ . To make the model more realistic, we can also consider that the qubits that are kept by  $A$  undergo some (although weaker) form of noise. Consider then similarly local depolarizing noise for these other qubits with noise level  $q$ . Now, if  $q \neq 1$  all three colors are affected by the noise and one then needs to actually carry out a 3-color purification protocol [21]. Although the procedure is still relatively robust (as we will see below when comparing to other protocols), the 3-colorability raises the noise threshold  $p$  for the transmitted qubits considerably even under very weak forms of noise  $q$  for the 5 qubits kept by  $A$ . In particular, for  $q = 0.99$  one finds that purification succeeds when  $p \geq 0.50$  while for  $q = 0.97$  the threshold is  $p \geq 0.56$ .

However, interestingly, to purify locally to an arbitrary 3-qubit state one does not need to be able to remotely prepare any 3-qubit state. It is enough to be able to prepare any state in the 3-qubit MES as given in the main text, and then transform it by LOCC to the desired state. Therefore, it is enough to have the ability to purify to the 6-qubit resource graph state that allows for remote preparation of the MES.

This graph state is also 3-colorable, but using again the invariants of [25] one can find that this state belongs to class 18, which includes 2-colorable graphs [23]. In fact, using the local complementation rule [23] it is easy to find the local unitaries that transform our graph to the one shown in Fig.2 b). One, therefore, just needs to ap-

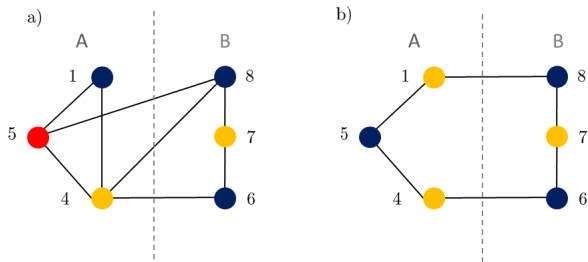


FIG. 2: *a)* This graphic shows the graph state corresponding to the resource state for REP of the MES. *b)* Via local complementation one obtains the representative of class 18.

ply the 2-color purification protocol [19]. This allows for much more robust purification when all qubits are subjected to noise. In this case, the corresponding thresholds are  $p \geq 0.44$  when  $q = 1$ ,  $p \geq 0.44$  when  $q = 0.99$  and  $p \geq 0.45$  when  $q = 0.97$ .

As we have already stressed, not only the ability to purify to any state is remarkable, but also the robustness of the scheme. To test this, consider the purification protocols for LMESs [17] and  $W$  states [20], which are the few procedures known in the literature besides graph states. For LMESs, we consider the 3-qubit regular LMES given by Eq. (5) when  $U$  is the 3-qubit  $\pi$ -phase gate, e.g.  $U = \mathbb{1} - 2|111\rangle\langle 111|$ . If the provider prepares this state and then sends it through the noisy channel,

the corresponding purification procedure works if the local noise for the three qubits is such that  $p \geq 0.81$  [17], which is noticeably worse than the threshold provided by the procedure we give here for both cases, using the 8-qubit and the 6-qubit resource graph states. The same happens in the case of the  $W$ , for which the best-known local depolarizing noise threshold is  $p \geq 0.69$  [20]. Moreover, the relative robustness of our scheme compared to these protocols is of fundamental character as the latter cannot be refined to achieve similar noise tolerances. To see this, consider the case of the  $W$  state. If one sends this state through the quantum channel given by (7), no purification is possible when the local noise level is such that  $p \leq 0.58$ . This is because, the output of this channel applied to the  $W$  state is PPT and, hence, non-distillable. Remarkably, our scheme would allow to accomplish the task under channels with such (and even considerably bigger) amounts of noise for the transmitted qubits.

Finally, notice that, of course, a similar purification scheme can be established using teleportation. For that,  $A$  would send one share of a bipartite maximally entangled state to each party in  $B$ . After the noisy transmission these pairs can be purified back to maximally entangled states and then  $A$  would simply teleport the target state. The purification to bipartite maximally entangled states succeeds whenever  $p > 1/3$  in our example (assuming no noise in the qubits held by  $A$ ), which offers a better noise tolerance than our REP-based scheme. However, notice that the teleportation strategy demands that the parties in  $A$  remain together (otherwise they cannot prepare and teleport the target state). On the other hand, the REP strategy only requires local operations from the parties in  $A$  and, hence, once the resource graph state has been distributed they can be spatially separated and still prepare any target state.